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On the existence of high Lewis number combustion fronts

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Abstract

We study a mathematical model for high Lewis number combustion processes with the reaction rate of the form of an Arrhenius law with or without an ignition cut-off. An efficient method for the proof of the existence and uniqueness of combustion fronts is provided by geometric singular perturbation theory. The fronts supported by the model with very large Lewis numbers are small perturbations of the front supported by the model with infinite Lewis number.
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Keywords: Geometric singular perturbation theory; Combustion fronts; Ignition cut-off

1. The model

We consider a well-known model for the propagation of combustion waves in the case of premixed fuel, with no heat loss, in one spatial dimension $x \in \mathbb{R}$. The system describing evolution of the temperature u and concentration of the fuel y reads

$$\begin{aligned} u_t &= u_{xx} + y\Omega(u), \\ y_t &= \varepsilon y_{xx} - \beta y\Omega(u). \end{aligned} \tag{1}$$

We will first describe the parameters of the system and then discuss the reaction terms. The system has two parameters. One is the exothermicity parameter $\beta > 0$ which is the ratio of the activation energy to the heat of the reaction. The other is the reciprocal of the Lewis number $\varepsilon = 1/\mathrm{Le} > 0$. Therefore, ε represents the ratio of the fuel diffusivity to the heat diffusivity. The system has been studied for various parameter regimes. Of interest to us are traveling wave solutions to (1) in two cases. One is the system (1) with $\varepsilon = 0$ ($\mathrm{Le} = \infty$). Its physical prototype is the combustion of solid fuels, more precisely, combustion that involves the solid phase only with no gaseous products present. The other is the case of $0 < \varepsilon \ll 1$, i.e., when Le is very large but finite. This situation is also physical: (1) then describes burning of very high density fluids at high temperatures.

The interest in the relation between cases with zero and nonzero ε is explained by two facts. First is that during the burning of solid fuels some liquefaction of the fuel might occur in the reaction zone, thus causing a nonzero value of

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$\varepsilon = 1/\mathrm{Le}$. The other is that the model for the combustion of solid fuels originated in physics as a singular limit of (1) as $\varepsilon \to 0$ (see [14] and the references therein).

The existence and properties of combustion waves for both cases have been studied, nevertheless the geometric approach that we will use to construct the wave allows to obtain information about rates of convergence of the wave to its rest state. That information is proved important for the further stability analysis of the wave [11,9].

The form of the reaction term depends on the way the cold boundary issue is resolved (see [3]).

One approach is to assume that the reaction starts at the ambient temperature. The reaction rate then has a form of an Arrhenius law without an ignition cut-off: $\Omega(u) = e^{-1/u}$, for $u > 0$ and $\Omega(u) = 0$ otherwise.

A different approach is to assume that the reaction starts at some positive ignition temperature [6,7], i.e. there exists $U_{ign} > 0$ such that $\Omega(u) = e^{-1/u}$, for $u \geq U_{ign}$ and $\Omega(u) = 0$ otherwise.

Of a special interest are waves of the front type, more precisely, fronts that asymptotically connect the state where the fuel is completely burnt, i.e., $y = 0$, and the maximal temperature $u = (1/\beta)$ is reached, and the state when none of the fuel is yet burnt and the temperature u is still zero:

$$(u, y) = (1/\beta, 0) \text{ at } -\infty, \quad \text{and} \quad (u, y) = (0, 1) \text{ at } \infty. \tag{2}$$

To find a traveling wave we introduce in (1) a moving coordinate frame $z = x - ct$,

$$\begin{aligned} u_t &= u_{zz} + cu_z + y\Omega(u), \\ y_t &= \varepsilon y_z + cy_z - \beta y\Omega(u). \end{aligned} \tag{3}$$

In the system above, with an abuse of notation, we think of u and y as functions of z, not x, t. The parameter c represents the speed of the wave. Traveling waves are sought as stationary solutions of (3),

$$\begin{aligned} &u'' + cu' + y\Omega(u) = 0, \\ &\varepsilon y'' + cy' - \beta y\Omega(u) = 0, \end{aligned} \tag{4}$$

where the derivative is with respect to z.

In the case without an ignition cut-off, for (4) with no additional assumptions on $\beta > 0$, and when $\varepsilon = 0$, the existence and uniqueness of a front that converges to both of its rest states exponentially fast has been proved in [5,18]. The existence of solutions with algebraic rates of decay to the equilibria was noticed, but these were considered to be of little interest [18]. The front has been observed numerically in [19]. When $\beta = 1$ and $\varepsilon = 0$ the existence and uniqueness of the fronts has been shown in [15] using methods similar to ones used in [4]. The existence of a front for any $\varepsilon > 0$ (and $\beta = 1$) has been also studied in [16]. It was also shown in [16] that a solution to the problem without an ignition cut-off can be obtained as a limit of the solutions of the problems with small ignition temperatures U_{ign} as $U_{ign} \to 0$. The relation between fronts $\varepsilon = 0$ and $0 < \varepsilon \ll 1$ is investigated in [1]: the dependence of the velocity on β is captured numerically, and the fronts are constructed using a geometric argument. In addition, the formula for the corrective term for the velocity of propagation of the perturbed front is obtained.

The case when the reaction rate is assumed to have a positive cut-off has also been studied. For $\varepsilon > 0$, a general existence result was obtained in [4] using a topological degree argument. The uniqueness is proved for the case $\varepsilon = 1$, and in [14] for $\varepsilon = 0$.

We consider the case of $0 < \varepsilon \ll 1$ for which the geometric singular perturbation theory seems to be a natural approach that, in addition, works similarly well for both versions of the nonlinearity, without (see [1]) and with an ignition cut-off. Besides the geometric construction of the front prepares grounds for the stability analysis of the front by providing information about the rates of the convergence of the front to its rest states and therefore gives an indication about the stability techniques to use.

The proof schematically can be described as follows. The system that represents the traveling wave equtions has a slow-fast structure. Moreover, there is a linear algebraic relation satisfied along any solution of (4), coming from an invariant of the equations,

$$\beta u'' + \beta cu' + \varepsilon y'' + cy' = 0.$$

Thus, the quantity $\beta u' + \beta cu + \varepsilon y' + cy$ is conserved along trajectories. The boundary conditions (2) are satisfied if

$$\beta u' + \beta cu + \varepsilon y' + cy = c. \tag{5}$$

It is also easy to see that both u and y that solve the traveling wave equation (4) are positive and monotone: u is decreasing and y is increasing.

In the case of $\varepsilon = 0$ the flow is restricted to a two-dimensional invariant manifold. The manifold is normally hyperbolic and attracting, therefore, by Fenichel's First theorem [8], it perturbs to an attracting manifold invariant for the flow with $\varepsilon > 0$. For the reduced problem, the lower dimension of the problem can be used to show that the front in the $\varepsilon = 0$ case is realized as a transversal intersection of relevant invariant manifolds. Transversality can be proved by a Melnikov integral calculation (see [1] for the case without an ignition cut-off), or by following the blueprint of the proof of the existence and uniqueness of subsonic detonation waves [10]. The proof follows the same logic as the proof in [2], where the existence and uniqueness of a traveling wave was proved for an equation and a system arising in a phase field and a generalized phase field models. For a discontinuous $\Omega(u)$ we will make adjustments to the argument above.

We state the results on the existence of the waves for small ε discussed above in the following theorem.

Theorem 1. *There exists $\varepsilon_0 > 0$ such that for each $0 \le \varepsilon \le \varepsilon_0$ system (1) has a unique, up to translation, traveling wave that satisfies boundary conditions (2) and does so at exponential rates. The speed $c = c(\varepsilon)$ of the wave is a continuous function of $\varepsilon \in [0,\varepsilon_0]$. As sets in the phase space, the orbits corresponding to $0 < \varepsilon < \varepsilon_0$* as $\varepsilon \to 0$ *converge to the unique orbit of the system with $\varepsilon = 0$.*

2. Traveling wave equation in the slow and fast scalings

We will first recall the properties of the components of the front.

Lemma 2. *Given the traveling wave (u,y) with $c > 0$ exists, it has the following properties: $y(z) > 0$ and $u(z) > 0$ for any z, and the wave is monotone in the sense that $y'(z) > 0$, $u'(z) < 0$ for any z.*

These properties of the front have been proved in [4] for $1 > \varepsilon > 0$ (with ignition temperature) and in [16] (without ignition temperature) and in [14] for $\varepsilon = 0$. Monotonicity of the front for $\varepsilon > 0$ case has been also proved in [17] under the assumption $y > 0$ and $u > 0$.

To prove Theorem 1 we rewrite the boundary value problem (4), (2) in an equivalent forms as follows. First, we rescale the spacial variable $z = \xi/c$,

$$\begin{aligned} &u'' + u' + \frac{1}{c^2} y\Omega(u) = 0, \\ &\varepsilon y'' + y' - \frac{\beta}{c^2} y\Omega(u) = 0, \end{aligned} \tag{6}$$

where, with an abuse of notation, the derivative is with respect to ξ. In the new variable the conserved quantity is

$$\beta u' + \beta u + \varepsilon y' + y = 1. \tag{7}$$

As a first order system (4) reads

$$\begin{aligned} &u' = v, \\ &v' = -v - \frac{1}{c^2} y\Omega(u), \\ &y' = z, \\ &\varepsilon z' = -z + \frac{\beta}{c^2} y\Omega(u). \end{aligned} \tag{8}$$

To match (2) we then seek an orbit of (8) such that

$$(u, v, y, z) \to \left(\frac{1}{\beta}, 0, 0, 0\right) \quad \text{at} \quad -\infty, \tag{9}$$

$$(u, v, y, z) \to (0, 0, 1, 0) \quad \text{at} \quad +\infty. \tag{10}$$

In the fast space variable $\eta = (\xi/\varepsilon)$, system (8) reads

$$\begin{aligned}
\dot{u} &= \varepsilon v, \\
\dot{v} &= \varepsilon \left(-v - \frac{1}{c^2} y\Omega(u) \right), \\
\dot{y} &= \varepsilon z, \\
\dot{z} &= -z + \frac{\beta}{c^2} y\Omega(u).
\end{aligned} \tag{11}$$

Here $\dot{}$ denotes differentiation with respect to η and $'$ stands for differentiation with respect to ξ.

The slow system (8), and the fast system (11) systems are equivalent for $\varepsilon \neq 0$. The problem amounts to finding a value of c for which a heteroclinic orbit satisfying (9)–(10) exists in either of these systems.

3. Construction of the orbit in $\varepsilon = 0$ case.

When $\varepsilon = 0$ in the system (8), the last equation reduces to an algebraic relation

$$z = \frac{\beta}{c^2}\Omega(u), \tag{12}$$

and the invariant manifold (5) is given by

$$y + \beta u + \beta v = 1. \tag{13}$$

On the manifold M_0 which is defined as the intersection of (13) and (12) the flow is two-dimensional,

$$\begin{aligned}
u' &= \frac{1}{\beta}(1 - \beta u - y), \\
y' &= \frac{\beta}{c^2} y\Omega(u),
\end{aligned} \tag{14}$$

Note here that smoothness of M_0 is determined by the smoothness of $\Omega(u)$.

We are looking for a heteroclinic orbit of (14) which connects $(u,y) = (1/\beta,0)$ to $(u,y) = (0,1)$. The point $(1/\beta,0)$ has two eigenvalues: $-1 < 0$ and $(\beta/c^2)e^{-\beta} > 0$, thus, is a saddle. Let $W^u(1/\beta,0)$ be its one-dimensional unstable manifold. The point (0,1) has eigenvalues 0 and -1, so there is a one-dimensional stable manifold $W^s(0,1)$ and one-dimensional center manifold $W^c(0,1)$. This is true for both continuous $\Omega(u)$ and the one with an ignition cut-off. The center manifold $W^c(0,1)$ is given in a neighborhood of (0,1) by the straight line $y = 1 - \beta u$, which in the second case consists entirely of equilibria.

We seek a heteroclinic orbit as an intersection of $W^u(1/\beta,0)$ and $W^s(0,1)$. Our claim can be stated as follows:

Theorem 3. *There is a unique value of* c_0, *for which system* (14) *has an orbit* (u,y) *satisfying boundary conditions* (2), *where the convergence to the equilibria is exponential.*

The proof for the case without an ignition cut-off can be found in [18], see also [12]. The idea for the existence proof is to show that for c very large $W^s(0,1)$ lies above $W^u(1/\beta,0)$ and for c sufficiently small $W^s(0,1)$ lies below $W^u(1/\beta,0)$, therefore there exists at least one value value of c for which the orbit exists.

The same arguments work for the case with an ignition cut-off: the $W^s(0,1)$ can be calculated exactly, since the system is linear in nearby the equilibrium. More precisely, for the linearized system

$$\begin{aligned}
u' &= -u, \\
y' &= 0,
\end{aligned}$$

the stable manifold is given by $y = 1$ which happens to be independent of c. The same way as in the previous case one can show that for very large values of c the set $W^u(1/\beta,0)$ lies below $y = 1$ and for very small values of c the set $W^u(1/\beta,0)$ lies above $y = 1$.

To prove the uniqueness, in the case without an ignition cut-off, the transversality of the intersection has been checked by a Melnikov type calculation. For the case with an ignition cut-off we find it more suitable to check the transversality directly: we will study the intersection of the stable and unstable manifolds precisely at $u = U_{ign}$ plane. This will be done in the proof of the next lemma. The same arguments work for the case without a cut-off, with the intersection of the stable and unstable manifolds considered at $u = U_0$ plane, with any $0 < U_0 < 1$.

First we extend (14) to the (u,y,c) space

$$\begin{aligned} u' &= \frac{1}{\beta}(1-\beta u - y),\\ y' &= \frac{\beta}{c^2} y\Omega(u),\\ c' &= 0. \end{aligned} \tag{15}$$

The critical points of this system are $(1/\beta,0,c)$ and $(0,1,c)$ for any c. We form a center-unstable manifold for $(1/\beta,0,c)$, denoted W^{cu}. On the other end, at $+\infty$, we are not interested in the full center-stable manifold. Rather we form a stable manifold for the submanifold of the center manifold of $(0,1,c)$ which does not include any points of $y = 1 - \beta u$ other than $u = 0$, $y = 1$. It is easy to see that this set is independent of c, it is given by $y = 1$ for all c. With an abuse of notation, we call this two-dimensional manifold W^{cs}.

Lemma 4. *When $\varepsilon = 0$, $W^{cu}(1/\beta,0,c_0)$ and $W^{cs}(0,1,c_0)$ intersect transversally.*

Proof. Case of an $\Omega(u)$ without an ignition cut-off. We consider the intersections of W^{cu} and W^{cs} with the plane $u = U_0$, $0 < U_0 < 1$. We denote the intersections (which are curves in the plane $u = U_0$) $Y = h^-(c)$ and $Y = h^+(c)$, respectively. The proof then is reduced to checking the transversality condition [13, Sect. 1.4]:

$$\left(\frac{\partial h^-}{\partial c} - \frac{\partial h^+}{\partial c}\right)\bigg|_{c=c_0} \neq 0. \tag{16}$$

We start by identifying vectors tangent to W^{cu} and W^{cs}. One of them is

$$H_1^{\pm} = \left(\frac{\partial u}{\partial c}, \frac{\partial y}{\partial c}, 1\right) = (\delta u^{\pm}, \delta y^{\pm}, 1). \tag{17}$$

Another one is the vector field of (15)

$$H_2 = (f_1(u, y, c), f_2(u, y, c), 0), \tag{18}$$

where $f_1(u,y,c) = (1/\beta)(1 - \beta u - y)$, $f_2(u,y,c) = (\beta/c^2)y\Omega(u)$. To find the sign of $(\partial h^-/\partial c)$ at $c = c_0$ we look at the following vector product

$$\begin{vmatrix} i & j & k \\ f_1 & f_2 & 0 \\ \delta u^{\pm} & \delta y^{\pm} & 1 \end{vmatrix} = f_2 i - f_1 j + (f_1\delta u^{\pm} - f_2\delta y^{\pm})k. \tag{19}$$

Note that on the plane $u = U_0$, we have $H_1^{\pm} = (0, (\partial h^{\pm}/\partial c), 1)$. The k th coordinate in (19) is

$$f_1\delta u^{\pm} - f_2\delta y^{\pm}|_{u=U_0} = f_2\frac{\partial y^{\pm}}{\partial c}. \tag{20}$$

Differentiating the quantity $w = f_1\delta u^{\pm} - f_2\delta y^{\pm}$ with respect to ξ we obtain

$$w' = \left(\frac{\partial f_1}{\partial u} + \frac{\partial f_2}{\partial y}\right) w + f_1\frac{\partial f_2}{\partial c} - f_2\frac{\partial f_1}{\partial c}.$$

Then w satisfies a differential equation

$$w' = \left(-1 + \frac{\beta}{c^2}\Omega(u)\right) w - \frac{2}{c^3}(1 - \beta u - y)y\Omega(u). \tag{21}$$

The properties obtained in Lemma 2 together with $1 - \beta u - y = \beta u' < 0$ yield that

$$F := -\frac{2}{c^3}(1 - \beta u - y)y\Omega(u) \geq 0.$$

The rates of decay of the front to its rest states, which are no slower than $e^{-\xi}$ at $+\infty$ and $e^{(\beta/c^2)e^{-\beta}\xi}$ at $-\infty$, allow us to integrate (21) from $-\infty$ to ξ and from ξ to $+\infty$. In the first case we obtain

$$w(z) = e^{-a_-(\xi)} \int_{-\infty}^{\xi} e^{a_-(s)} F(s)\,\mathrm{d}s, \tag{22}$$

where

$$a_-(\xi) = \left[1 - \frac{\beta}{c^2}e^{-\beta}\right]\xi - \frac{\beta}{c^2}\int_{-\infty}^{\xi}(\Omega(u(s)) - e^{-\beta})\,\mathrm{d}s.$$

Then (22) yields $w(0^-) > 0$. From (20) and $y > 0$ we conclude that

$$\left.\frac{\partial h^-}{\partial c}\right|_{c=c_0} < 0. \tag{23}$$

In the second case

$$w(z) = -e^{-a_+(\xi)} \int_{\xi}^{+\infty} e^{a_+(s)} F(s)\,\mathrm{d}s, \tag{24}$$

where

$$a_+(\xi) = \xi + \frac{\beta}{c^2}\int_{\xi}^{+\infty} e^{-\frac{1}{u(s)}}\,\mathrm{d}s.$$

Then (22) yields $w(0^+) < 0$ and thus from (20)

$$\left.\frac{\partial h^+}{\partial c}\right|_{c=c_0} > 0. \tag{25}$$

Eqs. (23) and (25) give

$$\left.\left(\frac{\partial h^-}{\partial c} - \frac{\partial h^+}{\partial c}\right)\right|_{c=c_0} < 0, \tag{26}$$

which completes the proof of the transversality condition (16).

The case of an $\Omega(u)$ with an ignition cut-off. We proceed the same way as above, but take $U_0 = U_{ign}$. Another difference in the proof is that the set W^{cs}, which is given by $y = 1$, is independent of c. Therefore in the transversality condition $(\partial h^+/\partial c)|_{c=c_0} = 0$, but $(\partial h^-/\partial c)|_{c=c_0} < 0$, so (26) still holds. □

We want to construct a manifold on which a perturbation of Eq. (8) governs the flow. This manifold will be invariant under the full Eq. (11) or, equivalently (8). Since the intersection that creates the heteroclinic in (15) is transverse, it will perturb to (8). The manifold in question is constructed as a perturbation of the critical manifold for the limiting ($\varepsilon = 0$) system.

We invoke the invariant relation (5), $y + \beta u + \beta v + \varepsilon z = 1$, to reduce (11)

$$\dot{u} = \varepsilon v,$$
$$\dot{v} = \varepsilon \left(-v - \frac{1}{c^2} y\Omega(u) \right),$$
$$\dot{y} = \varepsilon z,$$
$$\dot{z} = -z + \frac{\beta}{c^2} y\Omega(u),$$

to a three-dimensional system

$$\begin{aligned} \dot{u} &= \varepsilon \left(\frac{1}{\beta} - u - \frac{1}{\beta} y - \frac{\varepsilon}{\beta} z \right), \\ \dot{y} &= \varepsilon z, \\ \dot{z} &= -z + \frac{\beta}{c^2} y\Omega(u). \end{aligned} \tag{27}$$

When $\varepsilon = 0$, (27) reads

$$\begin{aligned} \dot{u} &= 0, \\ \dot{y} &= 0, \\ \dot{z} &= -z + \frac{\beta}{c^2} y\Omega(u). \end{aligned} \tag{28}$$

We will again discuss the case of the continuous $\Omega(u)$ first and then point out the differences with the discontinuous case.

The case of an $\Omega(u)$ without an ignition cut-off. The manifold $\mathcal{M}_0 = \{z \in R^3 : z = (\beta/c^2) y\Omega(u)\}$ consist of equilibria for the fast system (28). The linearization of the right-hand side of (28) about any point in $\mathcal{M}_0^-$ has eigenvalues: 0 (double) and -1. Therefore the two-dimensional set $\mathcal{M}_0$ are attracting and normally hyperbolic.

Under these conditions invariant manifold theory by Fenichel is applicable. More precisely, by Fenichel's First Theorem, see [8] or [13], the critical manifold $\mathcal{M}_0$, at least over compact sets, perturbs to an invariant set for (27) with $\varepsilon > 0$ but small. We call this set $\mathcal{M}_\varepsilon$. The distance between $\mathcal{M}_0$ and $\mathcal{M}_\varepsilon$ is of order ε. If ε is small enough, $\mathcal{M}_\varepsilon$ is also normally hyperbolic and attracting on the fast scale $\eta = \xi/\varepsilon$. On $\mathcal{M}_\varepsilon$ the flow on the slow scale ξ is determined by equations that are an $O(\varepsilon)$ perturbation of (8). Indeed, by Fenichel's First Theorem, $\mathcal{M}_\varepsilon$ is given by

$$z = \Gamma(y, u, \varepsilon) = \frac{\beta}{c^2} y\Omega(u) + O(\varepsilon). \tag{29}$$

The equations for the flow on $\mathcal{M}_\varepsilon$ are then given by the following system

$$\begin{aligned} u' &= \frac{1}{\beta} - u - \frac{1}{\beta} y - \frac{\varepsilon}{c^2} \Gamma(y, u, \varepsilon), \\ y' &= \Gamma(y, u, \varepsilon). \end{aligned} \tag{30}$$

According to [8] any invariant set for (27) which is sufficiently close to $\mathcal{M}_0$ is located on $\mathcal{M}_\varepsilon$. Therefore the equilibria $(u,y,z) = (1/\beta,0,0)$ and $(u,y,z) = (0,1,0)$ belong to $\mathcal{M}_\varepsilon$.

The existence and uniqueness of the heteroclinic orbit of interest will be proved by showing that there exists a unique value of $c = c(\varepsilon)$ at which the unstable manifold W^{cu} of $(u,y) = (1/\beta,0)$ intersects transversely the stable manifold W^{cs} of $(u,y) = (0,1)$ as the speed parameter c varies. This will be done on $\mathcal{M}_\varepsilon$ as a perturbation of the same construction on $\mathcal{M}_0$. The reduction of the problem to $\mathcal{M}_\varepsilon$ is valid because of the following result.

Lemma 5. *For sufficiently small $\varepsilon > 0$, any heteroclinic orbit of (30) connecting $(1/\beta,0,0)$ to $(0,1,0)$ must lie in $\mathcal{M}_\varepsilon$.*

Proof. In deriving the critical manifold $\mathcal{M}_0$ for the case $\varepsilon = 0$, we obtained that $\mathcal{M}_0$ is an attracting set.

The slow manifold $\mathcal{M}_\varepsilon$ exists for sufficiently small ε and is also attracting. Therefore $W^u(1/\beta,0,0)$ must lie in $\mathcal{M}_\varepsilon$ for ε sufficiently small. □

Next let us show that the structure of the flow at $\xi \to \pm\infty$ for small ε is similar to one for $\varepsilon = 0$. The linearization of the flow (30) on $\mathcal{M}_\varepsilon$ about (0,1) has eigenvalues: 0 , −1.

There is a one-dimensional stable manifold and one-dimensional center manifold and therefore is essentially the same as in $\varepsilon = 0$ case. We will seek an orbit approaching the (0,1) along the stable manifold in $\mathcal{M}_\varepsilon$.

Within $\mathcal{M}_0$ the equilibrium $(1/\beta,0)$ when $\varepsilon = 0$ is a saddle with a one-dimensional unstable manifold W^u and a one-dimensional stable manifold. This structure perturbs to the case $\varepsilon > 0$: within M_ε the eigenvalues are −1 and $\beta e^{-\beta}/c$. We thus want to follow W^u as c is varied in $\mathcal{M}_\varepsilon$.

It suffices then to prove that for every $0 < \varepsilon \ll 1$ there exists c_ε such that

$$W^{cu}(1/\beta, 0, c_\varepsilon) \pitchfork W^{cs}(0, 1, c_\varepsilon) \neq \varnothing. \tag{31}$$

Lemma 4 shows that (31) is true when $\varepsilon = 0$. The transversality condition (16) proved for the case $\varepsilon = 0$ also allows us to claim that for the perturbed problem the intersection persists (see [13]). Let $W^{cu}(1/\beta,0,c) \cup \{u = U_0\}$ and $W^{cs}(0,1,c) \cup \{u = U_0\}$ on the manifold $\mathcal{M}_\varepsilon$ be given by $y = h^-(c,\varepsilon)$ and $y = h^+(c,\varepsilon)$, respectively. An intersection point $(c_\varepsilon, y_\varepsilon)$ of h^+ and h^- is found by solving

$$y = h^-(c, \varepsilon),$$

$$y = h^+(c, \varepsilon)$$

for y and c as functions of ε. This system has a unique solution if the determinant

$$\det \begin{vmatrix} -1 & \dfrac{\partial h^-}{\partial c} \\ -1 & \dfrac{\partial h^+}{\partial c} \end{vmatrix} = \frac{\partial h^-}{\partial c} - \frac{\partial h^+}{\partial c}$$

is nonzero at $c = c_0$ and $\varepsilon = 0$. This condition coincides with (16). By the Implicit Function Theorem, (31) then holds when $\varepsilon > 0$ also but with a nearby c_ε replacing c_0. Theorem 1 then easily follows.

The case of an $\Omega(u)$ with an ignition cut-off. If we allow the reaction function Ω to be discontinuous at $u = U_{ign}$, then the c-dependent critical manifold M_ε is also possibly discontinuous along $u = U_{ign}$. It consists of two sets: M_ε^-, which corresponds to $u > U_{ign}$, and M_ε^+ for $u < U_{ign}$. The set $M_0^- = M_0 \cap \{u \geq U_{ign}\}$ is a compact manifold with a boundary. We assume that the flow (8) is modified in a standard way to make Fenichel's invariant manifold theory applicable, i.e., under the modified flow the manifold is overflowing invariant and compact. Then, for small $\varepsilon > 0$, there exists an overflowing invariant manifold M_ε^- that is a perturbation of M_0 of order ε.

Since M_ε^- is attracting, instead of Lemma 5 we have the following statement: the solutions satisfying (9) at $\xi = -\infty$ do so along the unstable manifold of the corresponding equilibria and thus belong to M_ε^-.

On the other hand the solutions converging to (0,1) are solutions of a linear system. We recall that solutions of the full system (8) exist and are monotone functions of the spacial variable ξ. The intersection of the set W^{cs} with $u = U_{ign}$ is a line $L_\varepsilon = \{v = -U_{ign},\ \varepsilon z + y = 1\}$. When $\varepsilon = 0$ the manifold L_0, which is independent of c, and W^{cu}, which lies on M_0^-, intersect transversally at some $c = c_0$. When $\varepsilon > 0$ small enough, the intersection will persist at $c_\varepsilon = c_0 + O(\varepsilon)$.

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