STABILITY ANALYSIS FOR COMBUSTION FRONTS TRAVELING IN HYDRAULICALLY RESISTANT POROUS MEDIA *

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Abstract. We study front solutions of a system that models combustion in highly hydraulically resistant porous media. The spectral stability of the fronts is tackled by a combination of energy estimates and numerical Evans function computations. Our results suggest that there is a parameter regime for which there are no unstable eigenvalues. We use recent works about partially parabolic systems to prove that in the absence of unstable eigenvalues the fronts are convectively unstable.

Key words. traveling waves, combustion fronts, Evans function, spectral stability, nonlinear stability, partly parabolic systems

1. Introduction.

This paper is devoted to the stability analysis of a traveling front in a combustion model. A combustion front is a coherent flame structure that propagates with a constant velocity. In a physical system it represents an intermediate asymptotic behavior of a nonstationary combustion wave [5]. To capture combustion waves, mathematical models are often cast as systems of partial differential equations (PDEs) posed on infinite domains, with multiple spatially homogeneous equilibria that correspond to cold states and completely burned states. Traveling fronts then are exhibited as a result of a competition between two different states of the system. To know the stability of a combustion front is of importance for the general understanding of the underlying phenomena, and has implications, for example, for chemical technology and fire and explosion safety. From the mathematical point of view, combustion models are of interest because they are formulated as systems of coupled nonlinear PDEs of high complexity and as such require refined mathematical techniques for their treatment.

We consider here a system that is obtained from a model for combustion in highly hydraulically resistant porous media [37]. The original model [6, 37] is considered to adequately capture the rich physical dynamics of the combustion in hydraulically resistant porous media and at the same time be approachable from the mathematical point of view. In particular, this model seems to be the simplest one that captures the transition between deflagration and subsonic detonation. Although of a great importance, the latter issue will not be addressed in this paper, but we refer the interested readers to a recent paper [7] and references within. The model consists of a partly parabolic system of two PDEs coupled through a nonlinearity. Partly parabolic systems are systems where some but not all quantities diffuse.

The existence and uniqueness of combustion front solutions of the system have been studied extensively [9,12,17,18]. The stability of the fronts has not been studied yet to the authors' knowledge. We present here an attempt to provide a stability analysis that would allow for physical interpretation of the results.

The first stage of the stability analysis is to find the spectrum of the operator obtained by linearization of the PDE system about the wave. To do so, we first find

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the essential spectrum through the standard techniques described in [20]. Concerning the point spectrum, we find a bound on the eigenvalues (see Theorem 5.3). Our analysis relies on Hölder's and Young's inequalities and is similar to the study performed in [39]. Once the bound is known, we perform a numerical computation of the Evans function [1,10,21,30,33,43]. The Evans function is an analytic function of the spectral parameter defined in some region of the complex plane. It was used for example in the context of nerve impulse equations [1,10,21,43], pulse solutions to the generalized Korteveg-de Vries, Benjamin-Bona-Mahoney and Boussinesq equations [30], multi-pulse solutions to reaction-diffusion equations [2,3], solutions to perturbed nonlinear Schrödinger equations [22, 23, 25], near integrable systems [26], traveling hole solutions of the one-dimensional complex Ginzburg-Landau equation near the Nonlinear-Schrödinger limit [24], and in the context of combustion problems [4, 13, 19]. The Evans function can be expressed in several different ways depending on the size of the stable and unstable manifolds of the linear system. In this article, the Evans function is expressed as the scalar product of a solution of the linear system with a solution of the adjoint system [10, 21, 30]. Numerical computations of the Evans function in this case was first performed by Evans himself [11] and later by other authors (see for example [31,32,38]). We perform a numerical computation of the Evans function and use it to show that there is a parameter regime for which there is no unstable discrete spectrum that would cause perturbations to the front to grow exponentially fast in time.

Absence of unstable discrete spectrum does not necessarily imply stability. This is due to the fact that (i) the front has continuous spectrum that is merely marginally stable, and (ii) the system is partially parabolic, so the linearized operator is not sectorial and the semigroup it generates is not analytic but only C^0. We overcome these issues by using recently obtained results about fronts with marginal spectrum and about partially parabolic systems.

The structure of the paper is as follows. We first introduce the full system, describe existing results related to the combustion fronts, and then explain how the reduced system is obtained. Then we analytically and numerically study the stability of the combustion fronts. To simplify our computations, we perform our study on the reduced system. We then show for the fronts as solutions of the full system, no new instabilities are created and our results concerning the reduced system carry over to the full system.

2. Background. In this paper we consider a system that is related to the combustion model for highly hydraulically resistant porous media [37]. The porous media combustion model was formulated in [6] and consists of a system of coupled PDEs

$$\begin{aligned} v_t - (1-\gamma^{-1})u_t &= \varepsilon v_{xx} + yF(v), \\ u_t - v_t &= u_{xx}, \\ y_t &= \varepsilon \mathrm{Le}^{-1} y_{xx} - \gamma y F(v), \end{aligned} \tag{2.1}$$

where v, u and y are scaled temperature, pressure, and concentration of the deficient reactant; $\gamma > 1$ is the specific heat ratio, ε is a ratio of pressure and molecular diffusivities, Le is a Lewis number, and $yF(v)$ is the reaction rate. The first and the third equations of the system (2.1) represent the partially linearized conservation equations for energy and deficient reactant, while the second one is a linearized continuity equation combined with Darcy's law. The original system of physical laws and the deduction of (2.1) can be found in [18].

It is assumed that function $F(v)$ is of the Arrhenius type with an ignition cut-off, that is,

$$F(v) = 0 \text{ for } 0 \leq v < v_{ign} < 1, \text{ and } F(v) > 0 \text{ for } v > v_{ign}, \tag{2.2}$$

and $F(v)$ is an increasing Lipschitz continuous function, except for a possible discontinuity at the ignition temperature $v = v_{ign}$. The presence of the ignition temperature reflects the fact that for low temperatures the reaction rate is many orders less than that at high temperatures. At the same time, forcing the reaction term to be zero at low temperatures is necessary for the existence of combustion fronts. It is also expected that the behavior of solutions depend on the value of the ignition temperature weakly, to a certain degree. We refer readers to [41] for details. We also point out that introducing a small parameter describing ignition temperature is typical for combustion models (see [27] for more examples).

In [40], for example, in the system (2.1) the reaction rate $F(v)$ is taken to be a nonlinear discontinuous function

$$F_{jump}(v) = \begin{cases} \exp\left(Z\left\{\frac{v-(1-\gamma^{-1})}{\sigma+(1-\sigma)v}\right\}\right), & v \geq v_{ign}, \\ 0, & v < v_{ign}. \end{cases} \tag{2.3}$$

Here $Z > 0$ is the Zeldovich number, and $0 < \sigma < 1$ is a ratio of the characteristic temperatures of fresh and burned reactant [16].

Initial conditions of the following form are usually considered [16], for the process of initiation of combustion,

$$v(0,x) = v_0(x), \quad y(0,x) = 1, \quad u(0,x) = 0. \tag{2.4}$$

For realistic materials ε varies in the range 10^{-2} - 10^{-5} [37] and therefore it can be treated as a small parameter. There is a numerical evidence that a small thermal diffusivity has no or very minor effect on details of propagation of the detonation wave [37]. Hence, (2.1) is often simplified by setting $\varepsilon = 0$,

$$\begin{aligned} v_t - (1-\gamma^{-1})u_t &= yF(v), \\ u_t - v_t &= u_{xx}, \\ y_t &= -\gamma y F(v). \end{aligned} \tag{2.5}$$

We note here that this simplification should be done cautiously as (i) this is a valid simplification only for materials for with Le is not small, (ii) in general, introducing the term $\varepsilon \mathrm{Le}^{-1} y_{xx}$ constitutes a singular perturbation of the system (2.5) and as such can produce drastic changes in the dynamics.

For both systems, of interest is the existence and uniqueness of traveling fronts that asymptotically connect the completely burned state to the state with all of the reactant present. More precisely, one seeks solutions of (2.1) or (2.5), that are solutions of the form $v(x,t) = v(\xi)$, $u(x,t) = u(\xi)$, $y(x,t) = y(\xi)$, where $\xi = x - ct$ and c is the a priori unknown front speed. In other words, these solutions are solutions of the traveling wave ODEs

$$\begin{aligned} -cv' + c(1-\gamma^{-1})u' &= \varepsilon v'' + yF(v), \\ u'' &= c(v' - u'), \\ cy' + \varepsilon \mathrm{Le}^{-1} y'' &= \gamma y F(v), \end{aligned} \tag{2.6}$$

or, in the case $\epsilon = 0$,

$$(2.7)\qquad \begin{aligned} -cv' + c(1-\gamma^{-1})u' &= yF(v), \\ u'' &= c(v' - u'), \\ cy' &= -\gamma y F(v), \end{aligned}$$

subject to the boundary conditions

$$(2.8)\qquad \begin{aligned} &v(-\infty) = 1, \quad u(-\infty) = 1, \quad y(-\infty) = 0, \\ &v(+\infty) = 0, \quad u(+\infty) = 0, \quad y(+\infty) = 1. \end{aligned}$$

Traveling wave solutions of (2.5) are well studied. In [9] the existence and uniqueness of the front is proved using Leray-Schauder degree theory and estimates on the speed c are obtained. In [17], based on an ODE analysis it is shown that there is a unique value of $c = c_0$ for which a front exists that asymptotically connects the burned state to the unburned state. Note that in both cases [9, 17], the condition $0 < v_{ign} < 1 - \gamma^{-1}$ is necessary for the front to exist. In [18] it is shown that the solutions of the system (2.6) converge to the solution of (2.7) as $\varepsilon \to 0$. In [12] it is shown that the solution of the system (2.7) perturbs to a unique solution of the system (2.6) at least for small ε. In part, this justifies the simplification of (2.6) to (2.7), but only if one studies processes occurring at speeds that are not slow (i.e., of order $O(1)$ or faster) as this is a local uniqueness result.

To our knowledge, for no parameter values, the stability of the traveling fronts for either of these systems has been studied.

It is obvious that the system (2.5) has a time-conserved quantity, using which we introduce a new variable

$$m = v - (1-\gamma^{-1})u - \gamma^{-1}(1-y),$$

and transform the system (2.5) to

$$(2.9)\qquad \begin{aligned} m_t &= 0, \\ u_t &= \gamma u_{xx} + \gamma y F(m + (1-\gamma^{-1})u + \gamma^{-1}(1-y)), \\ y_t &= -\gamma y F(m + (1-\gamma^{-1})u + \gamma^{-1}(1-y)). \end{aligned}$$

With initial conditions (2.4), $m(x,t) = v_0(x)$, but we instead shall consider this PDE system with such initial conditions that

$$(2.10)\qquad v(x,0) - (1-\gamma^{-1})u(x,0) - \gamma^{-1}(1-y(x,0)) = 0.$$

The system (2.9) then reduces to

$$(2.11)\qquad \begin{aligned} u_t &= \gamma u_{xx} + \gamma y F((1-\gamma^{-1})u + \gamma^{-1}(1-y)), \\ y_t &= -\gamma y F((1-\gamma^{-1})u + \gamma^{-1}(1-y)). \end{aligned}$$

One can scale γ into the time variable and with the abuse of notation ($t \to \gamma t$) obtain

$$(2.12)\qquad \begin{aligned} u_t &= u_{xx} + yF((1-\gamma^{-1})u + \gamma^{-1}(1-y)), \\ y_t &= -yF((1-\gamma^{-1})u + \gamma^{-1}(1-y)). \end{aligned}$$

We denote

$$h = 1 - \gamma^{-1} \tag{2.13}$$

so (2.12) reads

$$\begin{aligned} u_t &= u_{xx} + yF(hu + (1-h)(1-y)), \\ y_t &= -yF(hu + (1-h)(1-y)), \end{aligned} \tag{2.14}$$

In this article, we first study the stability of the fronts as solutions of the reduced system (2.14) that is obtained from (2.9) when the initial condition (2.10) are imposed. However, as we show in Section 8, when the condition (2.10) is relaxed, no additional instabilities are created and our results obtained for the reduced system still hold. Therefore, our stability analysis applies to the front solutions of the system (2.9) and, thus, of the system (2.5), without additional assumptions.

3. Explicit form for the discontinuous term. We consider the system (2.14) with $h \in (0,1)$. Part of our stability analysis is based on numerical calculations of the spectrum of the operator obtained by linearizing (2.14). Therefore we cannot work with a general form of the nonlinearity (2.2) but have to be more specific. We aim at working with the discontinuous F given in (2.3), which, in the case of the reduced system (2.14) and with the relation (2.13), becomes

$$F_{jump}(v) = \begin{cases} \exp\left(Z\left\{\frac{v-h}{\sigma+(1-\sigma)v}\right\}\right), & v \geq v_{ign}, \\ 0, & v < v_{ign}. \end{cases} \tag{3.1}$$

However, for the stability analysis of the front, it will be necessary to remove the discontinuity and consider a smooth F, which is defined like F_{jump} everywhere except for a small interval $(v_{ign}, v_{ign} + 2\delta)$ where the function is modified so as to go to zero in a smooth and monotonic fashion. One way this can be done is to consider

$$F_\delta(v) = \begin{cases} \exp\left(Z\left\{\frac{v-h}{\sigma+(1-\sigma)v}\right\}\right), & v \geq v_{ign} + 2\delta, \\ F_{jump}(v)\, H_\delta(v - v_{ign} - \delta), & v_{ign} \leq v < v_{ign} + 2\delta, \\ 0, & v < v_{ign}, \end{cases} \tag{3.2}$$

where H_δ is a smooth approximation of the Heaviside function H such that

$$\lim_{\delta \to 0^+} H_\delta = H, \; H_\delta(x) = 1, \;\text{ for }\; x > \delta, \;\; H_\delta(x) = 0, \;\text{ for }\; x < -\delta.$$

The limit above is understood in the distributional sense. A simple example of how this H_δ can be chosen is given by

$$H_\delta(x) = \frac{1}{1 + e^{\frac{4x\delta}{x^2 - \delta^2}}}, \;\text{ for }\; |x| < \delta. \tag{3.3}$$

While the stability analysis necessitates the use of a smooth version of F, the nonlinear stability analysis performed in Section 7 is valid for any δ. Furthermore, the numerical computations are done for small values of δ. More specifically, the value of δ in the numerical computations of Section 6 are chosen so that the front velocity of the smoothed system is reasonably close to the velocity in the discontinuous system. Furthermore, we discuss below the fact that the front solution of (2.14) with $F = F_\delta$ approaches the front solution when F is chosen to be F_{jump} as $\delta \to 0^+$. We can thus interpret the stability results of this paper to be valid when δ in (3.2) is very small.

We now want to prove that the front solution of (2.14) with $F = F_\delta$ approaches the front solution of the discontinuous system as $\delta \to 0^+$. The construction of the front solution in [17] is based on the existence and uniqueness of a smooth solution of a boundary value problem in which the dependent variable is y and the independent is u. Using the notation corresponding to the system (2.14), the boundary problem is obtained from the traveling wave reduction given below in (4.2) and is given by

$$\frac{dy}{du} = \frac{yF(hu + (1-h)(1-y))}{c(1-u-y)},$$

with boundary conditions $y\left(\frac{v_{ign}}{h}\right) = 1$, $y(1) = 0$. Between the boundary points, $u = \frac{v_{ign}}{h}$ and $u = 1$, the argument of F (given by $hu + (1-h)(1-y)$) varies monotonically from v_{ign} to 1. Given that F is Lipschitz continuous on the interval $[v_{ign}, 1]$ and given that F in addition depends on a parameter δ (such as suggested in (3.2) with H_δ given in (3.3)), smoothly, and in a way that the conditions in the theorems of [17] are satisfied (Lipschitz continuity for all $v \geq v_{ign}$, $F(v) = 0$ for $v < v_{ign}$, and $0 < v_{ign} < h$), then the front and the value of c for which it exists depend on δ in a continuous way.

4. Front Solution. We write the system (2.14) in variable $\xi = x - ct$ and t:

$$\begin{aligned} u_t &= u_{\xi\xi} + cu_\xi + yF(w), \\ y_t &= cy_\xi - yF(w), \end{aligned} \tag{4.1}$$

where, for brevity, $w \equiv hu + (1-h)(1-y)$. The front solution then becomes a stationary solution of (4.1), i.e. a solution of the dynamical system

$$\begin{aligned} &u_{\xi\xi} + cu_\xi + yF(w) = 0, \\ &cy_\xi - yF(w) = 0. \end{aligned} \tag{4.2}$$

We denote by $(u, y) = (\widehat{u}, \widehat{y})$ the front solution, which satisfies

$$(\widehat{u}, \widehat{y}) \to (1, 0) \text{ as } \xi \to -\infty, \text{ and } (\widehat{u}, \widehat{y}) \to (0, 1) \text{ as } \xi \to \infty. \tag{4.3}$$

In the next sections we study the stability of the front $(\widehat{u}, \widehat{y})$. The most important part of the stability analysis is to find the spectrum of the linear operator produced by the front.

5. Linearization. In this section, we analytically study the spectrum of the linear operator arising from the linearisation of (4.1) about the front solution. We will first find the essential spectrum and then find a bound on any eigenvalue with positive real part. We point out that neither the procedure of finding the essential spectrum nor the numerical calculations of the discrete spectrum are sensitive with respect the choice of δ in (3.2). Indeed, the computation of the continuous spectrum will be the same for any choice of δ in the smooth version of F. Furthermore, as discussed before, the choice of δ is expected to have little effect on the numerical computations if δ is chosen to be small enough. However, in order to have a linear operator that is smooth at every point along the front, we consider the smooth version.

The eigenvalue problem arising from the linearization of (4.1) about $(\widehat{u},\widehat{y})$ reads

$$\begin{aligned} \lambda p &= p_{\xi\xi} + cp_\xi + F_w(\widehat{w})\,\widehat{y}\,(hp-(1-h)q) + F(\widehat{w})q,\\ \lambda q &= cq_\xi - F_w(\widehat{w})\,\widehat{y}\,(hp-(1-h)q) - F(\widehat{w})q. \end{aligned} \tag{5.1}$$

Alternatively, this system can be written as $\mathcal{L}W = \lambda W$, $W = (p,q)^T$, where

$$\mathcal{L} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + c\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}\xi} + \begin{pmatrix}F_w(\widehat{w})\,\widehat{y}\,h & F_w(\widehat{w})\,\widehat{y}\,(1-h)+F(\widehat{w})\\ -F_w(\widehat{w})\,\widehat{y}\,h & -F_w(\widehat{w})\,\widehat{y}\,(1-h)-F(\widehat{w})\end{pmatrix}. \tag{5.2}$$

Generally speaking, a traveling wave is called spectrally stable in $L^2(\mathbb{R})$ if the spectrum of the linear operator $\mathcal{L}$ on $L^2(\mathbb{R})$ is contained in the half-plane $\{\mathrm{Re}\,\lambda \le -\nu\}$ for some $\nu > 0$ with the exception of a simple zero eigenvalue which is generated by the translational invariance.

If the discrete spectrum of the linear operator $\mathcal{L}$ in $L^2(\mathbb{R})$ is stable, i.e it is contained in the half-plane $\{\mathrm{Re}\lambda \le -\nu\}$ for some $\nu > 0$, except for a simple eigenvalue at 0, but the essential spectrum has nonempty intersection with the imaginary axis then the wave sometimes is called spectrally unstable due to essential spectrum.

The system (4.1) is an example of a partly parabolic system: the y-component does not diffuse. Partly parabolic systems or, as they are also called, partly dissipative [35], or partially degenerate [28] are not as well studied as parabolic problems. The stability analysis in parabolic problems is based on the fact that the linearized operators are sectorial and thus generate analytic semigroups [20]. In the case of the system (4.1), the linear operator $\mathcal{L}$ is not sectorial, so it generates a C^0 semigroup. Hence, unlike the case of sectorial operators, generally speaking, the spectral stability of the wave does not simply imply either the linear stability of the traveling wave, i.e., the decay of the solutions of the linearized system, or the orbital stability of the wave, as it does in case of sectorial operators [20]. Recall that a wave is called orbitally stable if a solution that starts near the wave stays close to the family of translates of the wave, in some norm not necessarily the same as the norm of the original space. If, in addition, it converges (in some norm) to a particular translate of the wave then the wave is called orbitally or nonlinearly stable with asymptotic phase.

In any case, the stability analysis of a front starts with finding its spectrum. The spectrum of a front consists of the isolated eigenvalues of finite multiplicity of $\mathcal{L}$ and the continuous part of the spectrum that is called essential.

5.1. Essential Spectrum. We define $\mathcal{L}^{\pm\infty} = \lim_{x\to\pm\infty}\mathcal{L}$ and compute them by inserting $\widehat{y} = 1$ and $\widehat{w} = 0$ into (5.2) for $\mathcal{L}^{+\infty}$ and $\widehat{y} = 0$ and $\widehat{w} = 1$ for $\mathcal{L}^{-\infty}$:

$$\mathcal{L}^{+\infty} = \begin{pmatrix}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + c\frac{\mathrm{d}}{\mathrm{d}\xi} & 0\\ 0 & c\frac{\mathrm{d}}{\mathrm{d}\xi}\end{pmatrix},\quad \mathcal{L}^{-\infty} = \begin{pmatrix}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + c\frac{\mathrm{d}}{\mathrm{d}\xi} & e^{(1-h)Z}\\ 0 & c\frac{\mathrm{d}}{\mathrm{d}\xi} - e^{(1-h)Z}\end{pmatrix}.$$

The boundaries of the essential spectrum of $\mathcal{L}$ are given by the spectra of constant coefficient operators $\mathcal{L}^{\pm\infty}$ (see [20], Theorem A.2, p. 140). We compute these spectra using Fourier analysis, which amounts to computing the value of λ for which

$$\det\left(\mathcal{L}^{\pm\infty}_\sigma - \lambda I\right) = 0, \tag{5.3}$$

where $\mathcal{L}^{\pm\infty}_\sigma$ are obtained from $\mathcal{L}^{\pm\infty}$ by replacing $\frac{\mathrm{d}}{\mathrm{d}\xi}$ with $i\,\sigma$. The set of curves defined by the equation (5.3)

$$\begin{aligned} &\{\lambda = -\sigma^2 + ci\sigma;\ \sigma\in\mathbb{R}\}\cup\{\lambda = ci\sigma;\ \sigma\in\mathbb{R}\},\\ &\{\lambda = -\sigma^2 + ci\sigma;\ \sigma\in\mathbb{R}\}\cup\{\lambda = ci\sigma - e^{(1-h)Z)};\ \sigma\in\mathbb{R}\} \end{aligned} \tag{5.4}$$

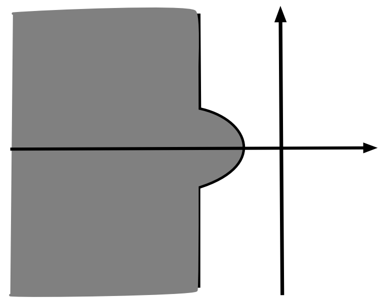

FIG. 5.1. *Essential spectrum in the space with an exponential weight.*

divides the complex plane into regions that are either covered by spectrum or, otherwise, contain only discrete eigenvalues. Here the first set is due to behavior at $+\infty$ and the second due to behavior at $-\infty$. It is therefore easy to see that the essential spectrum of $\mathcal{L}$ contains the imaginary axis as its rightmost boundary, as the open right-hand side of the complex plane can only contain point spectrum, i.e. isolated eigenvalues with finite multiplicity [36].

So on the spectral level the front is unstable at least due to the essential spectrum on the imaginary axis. We shall numerically show in the following sections that for some parameter values, the essential spectrum covering all of the imaginary axes is the only unstable spectrum that the front has. We test the essential spectrum using an exponential weight $e^{\alpha\xi}$ to see if at least on the spectral level the instability is convective. Indeed, introducing exponential weight in the eigenvalue problem by using new variable $(\widetilde{p},\widetilde{q}) = e^{\alpha\xi}(p,q)$ yields

$$(5.5)\qquad \begin{aligned} \lambda\widetilde{p} &= \widetilde{p}_{\xi\xi} + (c-2\alpha)\widetilde{p}_\xi + F_w(\widehat{w})\,\widehat{y}\,(h\widetilde{p}-(1-h)\widetilde{q}) + F(\widehat{w})\widetilde{q} + (-c\alpha+\alpha^2)\widetilde{p},\\ \lambda\widetilde{q} &= (c-2\alpha)\widetilde{q}_\xi - F_w(\widehat{w})\,\widehat{y}\,(h\widetilde{p}-(1-h)\widetilde{q}) - F(\widehat{w})\widetilde{q} - c\alpha\widetilde{q}. \end{aligned}$$

We use the constant coefficient operators ${\mathcal{L}_\alpha}^{\pm\infty}$ obtained from $\mathcal{L}^{\pm\infty}$ by replacing $\frac{\mathrm{d}}{\mathrm{d}\xi}$ with $i\,\sigma-\alpha$ and find the following curves for the boundaries of the essential spectrum:

$$(5.6)\qquad \begin{aligned} &\{\lambda = (i\sigma-\alpha)^2 + c(i\sigma-\alpha);\ \sigma\in\mathbb{R}\} \cup \{\lambda = c(i\sigma-\alpha);\ \sigma\in\mathbb{R}\},\\ &\{\lambda = (i\sigma-\alpha)^2 + c(i\sigma-\alpha);\ \sigma\in\mathbb{R}\} \cup \{\lambda = c(i\sigma-\alpha) - e^{(1-h)Z)};\ \sigma\in\mathbb{R}\}. \end{aligned}$$

The right most boundary $\lambda = -\sigma^2+\alpha^2-c\alpha+(c-2\alpha)i\sigma$ of this set, as a curve on the complex plane, is depicted on Fig. 5.1. An exponential weight $e^{\alpha\xi}$ with a rate such that $\alpha^2-c\alpha<0$ but $c-2\alpha>0$ pushes the curves described in (5.4) to the left of the imaginary axis. We formulate this fact as a lemma.

LEMMA 5.1. *Consider the system* (2.14) *with* $c = c(\gamma,h,\sigma)$ *such that a traveling front exists that the boundary conditions* (4.3) *are satisfied. Suppose* $0<\alpha<c/2$. *Then there exists* $\nu>0$ *such that the essential spectrum of* $\mathcal{L}$ *on* E_α *is contained in a half-plane* $\mathrm{Re}\,\lambda \le -\nu < 0$.

5.2. Bound on the point spectrum. We assume that λ is an eigenvalue of the operator $\mathcal{L}$ defined in (5.2) with $\mathsf{Re}\,(\lambda)>0$. We want to find a bound on the norm of λ in the complex plane. The argument we use here follows closely the argument presented in Section 3 of [39].

We first want to prove the following lemma, which provides a bound on λ.

LEMMA 5.2. *Assume λ is an eigenvalue with positive real part for the problem (5.1) on $L^2(\mathbb{R})$. Then*

$$|q(\xi)| \le h_2(\xi)\|p\|_{L^2}, \tag{5.7}$$

where

$$h_1(\xi) = \frac{1}{c}\int_\xi^\infty ((1-h)F_w(\widehat{w}(z))\widehat{y}(z) - F(\widehat{w}(z)))\,dz \tag{5.8}$$

and

$$h_2(\xi) = \frac{h}{c}\sqrt{\int_\xi^{+\infty} (F_w(\widehat{w}(z))\widehat{y}(z))^2 e^{-2(h_1(\xi)-h_1(z))}dz}. \tag{5.9}$$

Proof. We first observe that the expression defining F in (3.2) for $w \ge v_{ign}$ is an increasing function and thus F_w is nonnegative for all w, σ, and h in $(0,1)$. We now rewrite the second equation of (5.1) as a first order non-homogeneous linear equation for q:

$$cq_\xi + ((1-h)F_w(\widehat{w})\,\widehat{y} - F(\widehat{w}) - \lambda)\,q = h\,F_w(\widehat{w})\,\widehat{y}\,p,$$

which can be rewritten as

$$\left(q(\xi)\exp\left(-h_1(\xi) - \frac{\lambda}{c}\xi\right)\right)' = \frac{h}{c}F_w(\widehat{w}(\xi))\widehat{y}(\xi)\,\exp\left(-h_1(\xi) - \frac{\lambda}{c}\xi\right)p(\xi), \tag{5.10}$$

where h_1 is as in the lemma. Integrating both sides of (5.10) and solving for q gives

$$q(\xi) = -\frac{h}{c}\exp(h_1(\xi))\int_\xi^\infty F_w(\widehat{w}(z))\widehat{y}(z)\,\exp\left(-h_1(z) + \frac{\lambda}{c}(\xi - z)\right)p(z)\,\mathrm{d}z, \tag{5.11}$$

which exists because $\mathsf{Re}\,(\lambda) > 0$. Equation (5.11) implies

$$\begin{aligned}
|q(\xi)| &\le \frac{h}{c}\exp(h_1(\xi))\int_\xi^{+\infty} F_w(\widehat{w}(z))\widehat{y}(z)\exp(-h_1(z) + \frac{\lambda}{c}(\xi - z))p(z)dz \\
&\le \frac{h}{c}\exp(h_1(\xi))\int_\xi^{+\infty} F_w(\widehat{w}(z))\widehat{y}(z)\exp(-h_1(z))p(z)dz \\
&\le \frac{h}{c}\|p\|_{L^2}\sqrt{\int_\xi^{+\infty}(F_w(\widehat{w}(z))\widehat{y}(z))^2 e^{-2(h_1(\xi)-h_1(z))}dz},
\end{aligned}$$

where we have used Hölder's inequality to find the last expression. The last inequality gives (5.7). ◻

THEOREM 5.3. *If λ is an eigenvalue with positive real part, then we have the following inequalities:*

$$\mathsf{Re}\,(\lambda) \le h_3 + h_4 \quad \textit{and} \quad |\lambda| \le \frac{c^2}{4} + \sqrt{2}\,h_3 + h_4, \tag{5.12}$$

where $h_3 = \sqrt{\int_{\mathbb{R}} \left(F(\widehat{w}) + (1-h)F_w(\widehat{w})\,\widehat{y}\right)^2 h_2(\xi)^2\,d\xi}$ *and* $h_4 = h\sqrt{\int_{\mathbb{R}} \left(F_w(\widehat{w})\,\widehat{y}\right)^2\,d\xi}$.

Proof. We multiply the first equation of (5.1) by $\overline{p}$, then integrate and get

$$\lambda \int_{\mathbb{R}} |p|^2 \, \mathrm{d}\xi = c \int_{\mathbb{R}} p_\xi \overline{p} \, \mathrm{d}\xi - \int_{\mathbb{R}} |p_\xi|^2 \, \mathrm{d}\xi + \int_{\mathbb{R}} \left(F(\widehat{w}) - (1-h) F_w(\widehat{w}) \, \widehat{y}\right) \, q \, \overline{p} \, \mathrm{d}\xi + h \int_{\mathbb{R}} F_w(\widehat{w}) \, \widehat{y} \, |p|^2 \, \mathrm{d}\xi. \tag{5.13}$$

We take the real part of (5.13) to get

$$\mathsf{Re}\,(\lambda) \int_{\mathbb{R}} |p|^2 \, \mathrm{d}\xi = - \int_{\mathbb{R}} |p_\xi|^2 \, \mathrm{d}\xi + \int_{\mathbb{R}} \left(F(\widehat{w}) - (1-h) F_w(\widehat{w}) \, \widehat{y}\right) \, \mathsf{Re}\,(q \, \overline{p}) \, \mathrm{d}\xi + h \int_{\mathbb{R}} F_w(\widehat{w}) \, \widehat{y} \, |p|^2 \, \mathrm{d}\xi. \tag{5.14}$$

From (5.14), we obtain the inequality

$$\mathsf{Re}\,(\lambda) \int_{\mathbb{R}} |p|^2 \, \mathrm{d}\xi \leq \int_{\mathbb{R}} \left(F(\widehat{w}) + (1-h) F_w(\widehat{w}) \, \widehat{y}\right) \, |\, \mathsf{Re}\,(q \, \overline{p}) \,| \, \mathrm{d}\xi + h \int_{\mathbb{R}} F_w(\widehat{w}) \, \widehat{y} \, |p|^2 \, \mathrm{d}\xi.$$

Using the fact that $|\, \mathsf{Re}\,(q \, \overline{p}) \,| \leq |p \, q|$ and Lemma 5.2, the inequality above implies

$$\mathsf{Re}\,(\lambda) \, \|p\|_{L^2}^2 \leq \|p\|_{L^2} \int_{\mathbb{R}} \left(F(\widehat{w}) + (1-h) F_w(\widehat{w}) \, \widehat{y}\right) h_2(\xi) \, |p| \, \mathrm{d}\xi + h \int_{\mathbb{R}} F_w(\widehat{w}) \, \widehat{y} \, |p|^2 \, \mathrm{d}\xi.$$

The first part of (5.12) then follows from Hölder's inequality.

We now take the imaginary part of (5.13) to obtain

$$\mathsf{Im}\,(\lambda) \int_{\mathbb{R}} |p|^2 \, \mathrm{d}\xi = -\mathrm{i}c \int_{\mathbb{R}} p_\xi \overline{p} \, \mathrm{d}\xi + \int_{\mathbb{R}} \left(F(\widehat{w}) - (1-h) F_w(\widehat{w}) \, \widehat{y}\right) \mathsf{Im}\,(q \, \overline{p}) \, \mathrm{d}\xi. \tag{5.15}$$

We then add (5.14) and the modulus of (5.15) to obtain

$$(\mathsf{Re}\,(\lambda) + |\mathsf{Im}\,(\lambda)|) \int_{\mathbb{R}} |p|^2 \, \mathrm{d}\xi \leq c \int_{\mathbb{R}} |p_\xi| |\overline{p}| \, \mathrm{d}\xi - \int_{\mathbb{R}} |p_\xi|^2 \, \mathrm{d}\xi + h \int_{\mathbb{R}} F_w(\widehat{w}) \, \widehat{y} \, |p|^2 \, \mathrm{d}\xi + \int_{\mathbb{R}} \left(F(\widehat{w}) + (1-h) F_w(\widehat{w}) \, \widehat{y}\right) \left(|\mathsf{Re}\,(q \, \overline{p})| + |\mathsf{Im}\,(q \, \overline{p})|\right) \mathrm{d}\xi.$$

Next, we apply Young's inequality to get $c \, |p_\xi||p| \leq \frac{c^2 |p|^2}{4} + |p_\xi|^2$. Using $|\lambda| \leq \mathsf{Re}\,(\lambda) + |\mathsf{Im}\,(\lambda)|$, $|\mathsf{Re}\,(q \, \overline{p})| + |\mathsf{Im}\,(q \, \overline{p})| \leq \sqrt{2} |p||q|$, Lemma 5.2 and Hölder's inequality, we prove the second part of (5.12).

□

6. Numerical Computations. Note that for all the numerical computations presented in this section, the function F is chosen to be in the form (3.2). We use the approximation of the Heaviside function to be the one given in (3.3).

6.1. Front Solution. In order to compute the front solution of (4.2) numerically, we perform the following manipulations. We add the two equations of (4.2) together and integrate. We obtain the following system of equations

$$\begin{aligned} & u_\xi + cu + cz = r \\ & y_\xi - \frac{1}{c} y F(hu + (1-h)(1-y)) = 0, \end{aligned} \tag{6.1}$$

where r is constant. Since front solution $(\widehat{u}, \widehat{v})$ converges to (0,1) as $\xi \to \infty$, we set $r = c$. We are thus left with the following two-dimensional dynamical system to integrate:

$$u_\xi = c(1 - y - u)$$
$$y_\xi = \frac{1}{c} yF\left(hu + (1-h)(1-y)\right). \tag{6.2}$$

We are interested in the heteroclinic orbit that connects the the fixed points $(1, 0)$ and $(0, 1)$. The unstable direction of the point (1,0) is given by $(c^2, -c^2 - e^{(1-h)Z})$ with corresponding eigenvalue $e^{(1-h)Z}/c$. To find the front solution for given values of h, σ, v_{ign}, δ, and Z, we use a simple shooting method and make incremental changes in the value of c until an appropriate connection to the fixed point (0,1) is found. Figure 6.1 shows how the speed c varies as a function of the various parameters. In particular, Figure 6.1 (c) shows that the range of "reasonable" values for the unphysical parameter δ, i.e. values of δ for which the speed c is close to the speed of the discontinuous model, depends on σ. We use this graph as a guide on how to choose a reasonable value of δ. For this reason, in the subsequent computations in the article, we use $\delta = 0.0005$. We also have checked that such a value of δ is appropriate if we change the value of Z to 4 or 5 and keep the other parameters the same. By appropriate here we mean that in all the cases treated in this article, the percentage difference between the velocities of the smoothed and the discontinuous models do not exceed 2%.

In the case where $h = 0.3$, $Z = 6$, $\sigma = 0.25$, $v_{ign} = 0.01$, and $\delta = 0.0005$, it is found that $c = 1.4683$. Figure 6.2 shows the front solution as a function of ξ in that case.

6.2. Spectral Stability. We are now interested in numerically studying the point spectrum of the eigenvalue problem (5.1) on $L^2(\mathbb{R})$. The system (5.1) can be turned into a linear dynamical system of the form

$$X' = A(\xi, \lambda)\, X, \tag{6.3}$$

where A is the 3×3 following square matrix

$$A(\xi, \lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda - h\, F_w(\widehat{w})\, \widehat{y} & -c & (1-h)F_w(\widehat{w})\, \widehat{y} - F(\widehat{w}) \\ -h\, F_w(\widehat{w})\, \widehat{y}/c & 0 & \lambda/c + (1-h)\, F_w(\widehat{w})\, \widehat{y}/c + F(\widehat{w})/c \end{pmatrix}. \tag{6.4}$$

The asymptotic behavior as $\xi \to \infty$ of the solutions to (6.3) is determined by the matrix $A^\infty(\lambda) = \lim_{\xi \to \infty} A(\xi, \lambda)$, which is found by inserting the values $\widehat{y} = 1$ and $\widehat{w} = 0$ into (6.4)

$$A^\infty(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda & -c & 0 \\ 0 & 0 & \lambda/c \end{pmatrix}.$$

For $\mathsf{Re}\,(\lambda) > 0$, the unique eigenvalue of A^∞ with negative real part and the corresponding eigenvector are given by $\mu_+ = \frac{-c-\sqrt{c^2+4\lambda}}{2}$, $v_+ = (1,\ \mu_+\ ,0)^T$. System (6.3) then has a unique solution X_+ satisfying $\lim_{\xi \to \infty} X_+ e^{-\mu_+\, \xi} = v_+$ [8]. In this situation where the linear system (6.3) has a one-dimensional stable manifold at ∞, the definition of the Evans function requires the adjoint system [30]

$$Y' = -A^T(\xi, \lambda)\, Y. \tag{6.5}$$

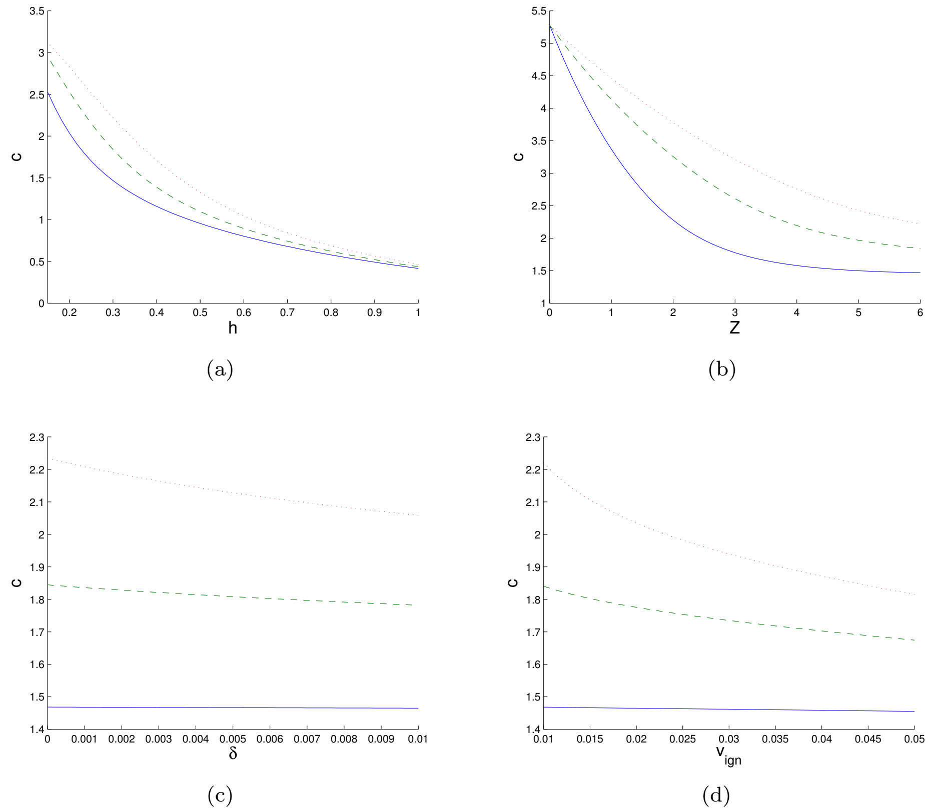

Fig. 6.1. *Plots showing the speed c of the front as a function of the parameters (a) h (with $Z = 6$, $v_{ign} = 0.01$, and $\delta = 0.0005$), (b) Z (with $h = 0.3$, $v_{ign} = 0.01$, and $\delta = 0.0005$), (c) δ (with $h = 0.3$, $Z = 6$, and $v_{ign} = 0.01$), and (d) v_{ign} (with $h = 0.3$, $Z = 6$, and $\delta = 0.0005$). In each of the four graphs, the solid line corresponds to $\sigma = 0.25$, the dashed line to $\sigma = 0.5$, and the dotted line to $\sigma = 0.75$.*

The asymptotic behavior as $\xi \to -\infty$ of the solutions to (6.5) is determined by the matrix $A^{-\infty}(\lambda) = \lim_{\xi\to-\infty} A(\xi, \lambda)$, which is found by inserting the values $\widehat{y} = 0$ and $\widehat{w} = 1$ into (6.4)

$$A^{-\infty}(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda & -c & -e^{(1-h)Z} \\ 0 & 0 & (\lambda + e^{(1-h)Z})/c \end{pmatrix}.$$

For $\mathsf{Re}\,(\lambda) > 0$, the unique eigenvalue of $-(A^{\infty})^T$ with positive real part and the corresponding eigenvector are given by

$$\mu_- = -\mu_+ = \frac{c + \sqrt{c^2 + 4\lambda}}{2}, \quad v_- = \left(-\lambda,\ \mu_-,\ \frac{c(c + \sqrt{c^2 + 4\lambda})}{ce^{Z(h-1)}(c + \sqrt{c^2 + 4\lambda} + 2\lambda/c) + 2} \right)^T.$$

System (6.5) then has a unique solution Y_- satisfying $\lim_{\xi\to-\infty} Y_- e^{-\mu_- \xi} = v_-$. The Evans function $D(\lambda)$ can then be defined as [30]

$$D(\lambda) = X_+(0) \cdot Y_-(0).$$

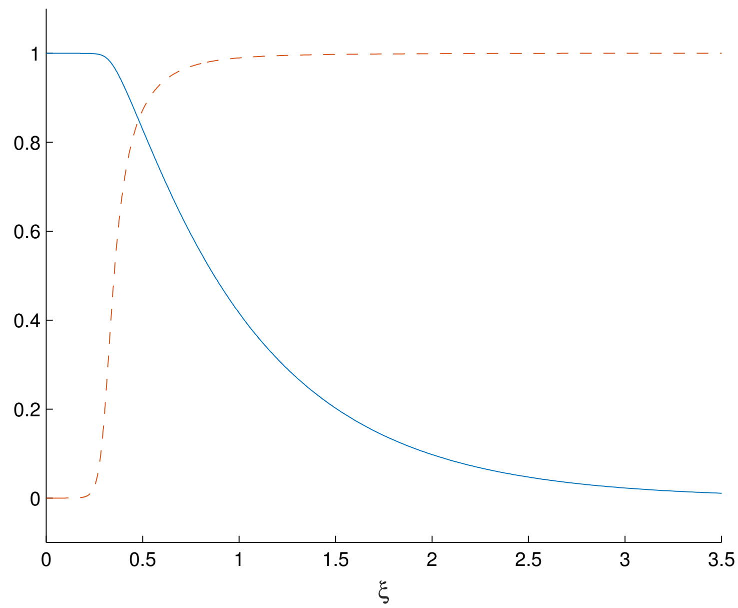

FIG. 6.2. *Front solution of the system (6.2) in the case $c = 1.4683$, $h = 0.3$, $Z = 6$, $\sigma = 0.25$, $v_{ign} = 0.01$, and $\delta = 0.0005$. The solid line corresponds to u and the dashed line to y.*

The Evans function is an analytic function in any region of the complex plane where μ_+ is the eigenvalue of A^∞ that has the smallest real part. In the region where μ_+ is the unique eigenvalue with negative real part (which, in our case, is the open right half of the complex plane), the zeroes of the Evans function correspond to the point spectrum of (5.1) on $L^2(\mathbb{R})$. Note that the three eigenvalues of A^∞ are $-c/2 \pm \sqrt{c^2+4\lambda}/2$ and λ/c, while the eigenvalues of $-(A^\infty)^T$ are $c/2 \pm \sqrt{c^2+4\lambda}/2$ and $-(\lambda + e^{Z(1-h)})/c$. This implies that to define a region where the Evans function is analytic, it suffices to consider the values of λ satisfying

$$\mathsf{Re}\,(\lambda) > -c^2/4. \tag{6.6}$$

To compute the Evans function numerically, we choose a positive value $\xi = L$ at which the matrix given in (6.4) is suitably close to its asymptotic value A^∞. We then integrate the system (6.3) backward from $\xi = L$ in the direction of the eigenvector v_+ and find $X_+(0)$. In order to eliminate the exponential growth due to the eigenvalue with negative real part as we integrate from $\xi = L$, we modify the system in the following way

$$X' = (A - \mu_+ I)\ X$$

and use the initial condition $X(L) = v_+$. Similarly, we find $Y_-(0)$ by integrating the adjoint system (6.5). Note that in the numerical computations, we choose the eigenvectors $v_\pm$ so that $v_+ \cdot v_- = 1$. This choice has the convenient consequence that $\lim_{|\lambda|\to\infty} D(\lambda) = 1$ [30].

In our case, since $F(w) = 0$ for $w < v_{ign}$, the matrix A in (6.4) reaches the constant matrix A^∞ for a finite positive value of ξ. When computing X_+, we thus choose the value of L to be the lowest value of ξ satisfying $\widehat{w}(\xi) < v_{ign}$.

Since we are interested in the zeroes of the Evans function, the standard method is to compute the integral of the logarithmic derivative of the Evans function on a given closed curve and obtain the winding number of $D(\lambda)$ along that curve. The eigenvalue problem (5.1) has an eigenvalue at $\lambda = 0$ due to the translation invariance of the system (2.14). In order to numerically verify that there are no other zeroes of the Evans function on the right side of the complex plane, we use the bound provided by Theorem 5.3. Once the quantities h_3 and h_4 are computed numerically, one can choose a closed curve whose points satisfy (6.6) and whose interior encloses the intersection of the right side of the complex plane and the region defined by (5.12). For example, in the case $h = 0.3$, $Z = 6$, $\sigma = 0.25$, $v_{ign} = 0.01$, and $\delta = 0.0005$ we find h_3 and h_4 to be, respectively, 140.0189 and 4.6702, and c to be 1.4683. Our numerical winding number computation then shows that the Evans function has no other zeroes than the one at the origin. Note that we can reduce our numerical computations by using the fact that the Evans function satisfy the property that $D(\overline{\lambda}) = \overline{D(\lambda)}$. Indeed, if we choose our contour of integration to be symmetric with respect to the real axis, it is sufficient to perform our numerical integration along the upper (or lower) half of the contour to determine the winding number.

We have made similar computations for other values of the parameters h, Z, σ, and v_{ign}. Specifically, we have considered an array of values of h and σ between 0 and 1 for $Z = 4, 5, 6$, while keeping $v_{ign} = 0.01$ and $\delta = 0.0005$. Every time, we have found the eigenvalue at the origin to be the only one. This thus strongly suggests that there is a regime in which the front solution is spectrally stable.

To study the possible presence of instability, it is useful to mention the fact that the system (2.14) is related to a well-studied combustion model. When $h = 1$, $\sigma = 0$, and $v_{ign} = 0$, the system (2.14) with F given by (3.1) can be scaled to the model for gasless combustion. Indeed, if one uses the change of variables $\widetilde{u} = u/Z$, $\widetilde{y} = y$, $t = (Z/e^Z)\widetilde{t}$, and $x = \sqrt{Z/e^Z}\,\widetilde{x}$, then system (2.14) becomes

$$
\begin{aligned}
u_t &= u_{xx} + ye^{-1/u}, \\
y_t &= -Zye^{-1/u},
\end{aligned} \tag{6.7}
$$

where we have suppressed the use of the tildes above the variables. System (6.7) is the model for gasless combustion with Z now playing the role of the exothermicity parameter (usually denoted by β) [4,5,19,39]. In this case, there is an instability that occurs at about $Z = 6.5$ [13,19,42]. This instability is caused by a pair of conjugate eigenvalues crossing the imaginary axis to the right side of the complex plane when Z reaches approximatively 6.5 [13,19]. As an example, we consider the following set of parameter values close to the gasless model: $h = 1$, $\sigma = 0$, $v_{ign} = 0.01$ and $\delta = 0.0005$. In this case, we find a pair of eigenvalues $\lambda = \pm 0.1765\mathrm{i}$ when $Z = 6.572$. These two eigenvalues move to the right side of the complex plane when Z is increased. We plan a study of this instability, for various values of the parameters in a future publication.

7. Interpretation of spectral information. In this section we use the information about the spectrum obtained above to describe the nature of the instability of the front in the case when the front does not have unstable discrete spectrum, i.e. the instability is caused by the continuous spectrum only.

Let E_0 denote H^1 or BUC, with norm $\| \ \|_0$. Let E_α be the corresponding weighted space with a weight $e^{\alpha\xi}$: $u \in E_\alpha$ if and only if $e^{\alpha\xi}u(\xi) \in E_0$, and $\|u\|_\alpha = \|e^{\alpha\xi}u(\xi)\|_0$.

An instability that can be removed by an exponential weight is called convective [33,34]. Indeed, if a perturbation grows while being convected to, for example, $-\infty$, then the perturbation may stay bounded or even decay in an exponentially weighted norm with a weight that decays at $-\infty$.

A wave that is unstable in the space E_0 but stable in some weighted space E_α, in the literature is called convectively unstable. Convective instability of a wave is characterized by the fact that small perturbations to the wave decay pointwise.

Since the system (2.14) is partially parabolic, the operator $\mathcal{L}$ of the linearization of this system about the front is not sectorial and generates not an analytic but a C^0-semigroup. To interpret the absence of the unstable discrete spectrum for the full nonlinear system, one needs appropriate estimates on the semigroup. The semigroup estimates follow from the main theorem in [15]. For convenience we make the theorem available to the reader in Section 9.1 of the Appendix.

It is important to note that for the system (2.14), the assumption of Theorem 9.1 holds in the space E_α (see Lemma 5.1).

In the weighted space the nonlinearity is not well defined, therefore the semigroup estimates (9.1) are not sufficient to simply imply nonlinear stability. Nevertheless, the system (2.14) and the front that it supports as a solution have properties that allow us to obtain nonlinear stability. The following statement follows from Theorem 3.14 in [14] which is formulated in Section 9.2 of the Appendix.

THEOREM 7.1. *Consider the system* (4.1) *with* $c = c(h, Z, \gamma, \sigma)$ *being the speed of the front* $\widehat{Y}(\xi) = (\widehat{u}(\xi), \widehat{y}(\xi))$ *and* h, Z, γ, σ *be such that the Evans function for the traveling wave* $\widehat{Y}(\xi)$ *has no zeros in the half-plane* $\operatorname{Re}\lambda \geq 0$ *other than a simple zero at the origin. Let* α *be chosen as in Lemma* 5.1. *Suppose the initial conditions for* (2.14) *be* $(\widehat{u}(\xi) + \widetilde{u}^0(\xi), \widehat{y}(\xi) + \widetilde{y}^0(\xi))$ *with* $(\widetilde{u}^0(\xi), \widetilde{y}^0(\xi)) \in E_\alpha^2$ *and* $\|(\widetilde{u}^0, \widetilde{y}^0)\|_\alpha$ *small, and let* $(u(t,\xi), y(t,\xi))$ *be the solution of* (2.14) *with* $(u(0,\xi), y(0,\xi)) = (\widehat{u}(\xi) + \widetilde{u}^0(\xi), \widehat{y}(\xi) + \widetilde{y}^0(\xi))$. *Then:*

1. $(u(t,\xi), y(t,\xi))$ *is defined for all* $t \geq 0$.
2. $(u(t,\xi), y(t,\xi)) = (\widehat{u}(\xi - q(t)) + \widetilde{u}(t,\xi), \widehat{y}(\xi - q(t)) + \widetilde{y}(t,\xi))$ *with* $(\widetilde{u}(t,\xi), \widetilde{y}(t,\xi))$ *in a fixed subspace of* E_α^2 *complementary to the span of* $(\widehat{u}', \widehat{v}')$.
3. $\|(\widetilde{u}(t), \widetilde{y}(t))\|_0 + |q(t)|$ *is small for all* $t \geq 0$.
4. $\|(\widetilde{u}(t), \widetilde{y}(t))\|_\alpha$ *decays exponentially as* $t \to \infty$.
5. *There exists* q^* *such that* $|q(t) - q^*|$ *decays exponentially as* $t \to \infty$.
6. *There is a constant* $C > 0$ *independent of* $(\widetilde{u}^0, \widetilde{y}^0)$ *such that* $\|\widetilde{u}(t)\|_0 \leq C\|(\widetilde{u}^0, \widetilde{y}^0)\|_\alpha$ *for all* $t \geq 0$.
7. $\|\widetilde{y}(t)\|_0$ *decays exponentially as* $t \to \infty$.

The stability results of Theorem 3.14 from [14] are applicable to the front in (4.1) because of Lemma 5.1 and the following properties of (4.1)

(i) $R(u, y_-) = R(u, 0) = \begin{pmatrix} yF(hu + (1-h)(1-y)) \\ -yF(hu + (1-h)(1-y)) \end{pmatrix}\Bigg|_{(u,0)} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where R is the nonlinearity in (4.1) and $y_- = 0$ is the y-component of the equilibrium to which the front converges at $-\infty$ (compare with Hypothesis 9.2 from Appendix).

(ii) The operator $\partial_{\xi\xi} + c\partial_\xi$ on E_0 generates a bounded semigroup.

(iii) The spectrum of the operator $c\partial_\xi - \mathrm{e}^{(1-h)Z}$ on E_0 is contained in $\operatorname{Re}\lambda \leq -\mathrm{e}^{(1-h)Z} < 0$ (compare with Hypothesis 9.3 from Appendix).

Under additional assumptions on the initial conditions, Theorem 3.16 in [14] (see Section 9.2) implies some more conclusions:

THEOREM 7.2. *Consider the Banach space* $E_0 \cap L^1(\mathbb{R})$ *with the norm* $\|u\|_{E_0 \cap L^1(\mathbb{R})} = \max\{\|u\|_{E_0}, \|u\|_{L^1(\mathbb{R})}\}$. *Suppose* $(\widetilde{u}^0(\xi), \widetilde{y}^0(\xi)) \in (E_\alpha \cap L^1(\mathbb{R}))^2$ *has norms in* E_α *and* L^1 *which are sufficiently small, and as in Theorem* 7.1, *let* $(u(t,\xi), y(t,\xi))$ *be the* *solution of* (2.14) *with* $(u(0,\xi),y(0,\xi)) = (\widehat{u}(\xi)+\widetilde{u}^0(\xi),\widehat{y}(\xi)+\widetilde{y}^0(\xi))$. *Then for all* $t \geq 0$*:*

1. $(\widetilde{u}(t,\xi),\widetilde{y}(t,\xi)) \in (E_\alpha \cap L^1(\mathbb{R}))^2$.
2. *There is a constant* C *independent of* $(\widetilde{u}^0(\xi),\widetilde{y}^0(\xi))$ *such that*

$$\begin{aligned}\|\widetilde{y}(t)\|_{L^1} &\leq C\ \max\left\{\|\widetilde{y}^0\|_{L^1}, \|(\widetilde{u}^0(\xi),\widetilde{y}^0(\xi))\|_\alpha\right\},\\ \|\widetilde{y}(t)\|_{L^\infty} &\leq C\ \min\{1,t^{-\frac{1}{2}}\}\ \max\left\{\|\widetilde{y}^0\|_{L^1}, \|(\widetilde{u}^0(\xi),\widetilde{y}^0(\xi))\|_\alpha\right\}.\end{aligned}$$

3. $\|\widetilde{u}\|_{L^1}$ *decays exponentially as* $t \to \infty$.

Theorem 7.1 implies that small perturbations to the concentration of the fuel which are far from the unburned state decay exponentially fast, while perturbations to the pressure and temperature stay bounded or, under additional assumptions (Theorem 7.2), decay at algebraic rates. These results have clear physical meanings: if some fuel is added in the area where the temperature is high, the fuel will burn out; if a cold or hot spot is introduced where there is no fuel left, the temperature will diffuse at algebraic rates as expected from the heat equation.

8. Conclusions and extension of the results. In this article, we have considered front solutions to the model (2.5). The fronts were seen as solutions of a reduced system (2.14) obtained by from (2.5) by scaling of the independent variables and requiring the initial conditions (2.10). We have used a combination of energy estimates computations and numerics based on the Evans function to show that there is a parameter regime for which there are no unstable eigenvalues. We then used recently obtained results about partially parabolic systems and about fronts with marginal spectrum to prove the nonlinear stability of the fronts in the absence of unstable spectrum. Our results are concerned with the case where the nonlinearity F takes the form (3.2) with δ very small, i.e. very close to the discontinuous version (3.1).

We now show that our spectral stability results and the nonlinear stability result described in Theorem 7.1 extend to the general case where the condition (2.10) does not necessarily hold, i.e. it applies to the fronts as solutions of (2.5). Recall that the system (2.5) is

$$\begin{aligned}m_t &= 0,\\ u_t &= u_{xx} + yF(m+hu+(1-h)(1-y)),\\ y_t &= -yF(m+hu+(1-h)(1-y)),\end{aligned} \tag{8.1}$$

where $m = v - hu - (1-h)(1-y)$. Under the initial condition (2.10), we have that $m = 0$ and the system (8.1) becomes the reduced system (2.14). To show that our stability results concerning the reduced system carry over to the general case (8.1), we first write the eigenvalue problem arising from the linearization of (8.1) about the front solutions

$$\begin{aligned}\lambda r &= cr_\xi,\\ \lambda p &= p_{\xi\xi} + cp_\xi + F_w(\widehat{w})\,\widehat{y}\,(r+hp-(1-h)q) + F(\widehat{w})q,\\ \lambda q &= cq_\xi - F_w(\widehat{w})\,\widehat{y}\,(hp-(1-h)q) - F(\widehat{w})(q-r),\end{aligned} \tag{8.2}$$

where we have used the fact that, in view of the asymptotic values for the front (4.3), we have $\widehat{m} \equiv \widehat{v} - h\widehat{u} - (1-h)(1-\widehat{y}) = 0$. The equation above generalizes the eigenvalue problem (5.1) obtained for the reduced system. Now, if λ is in the point spectrum, the first equation of (8.2) implies that $r = 0$, thus reducing the problem to the one we studied for the reduced system. Therefore, the point spectra of (5.1) and (8.2) are the same. As for the essential spectrum, if we apply the procedure described in Section 5.1 to (8.2), it is easy to see that the spectrum is unchanged with the exception that the presence of the equation for r adds one more covering of the imaginary axis.

The nonlinear stability results similar to one described in Theorem 7.1 can also be obtained from Theorem 9.4 and 9.5, since the diagonal operator

$$L^{(1)} = \begin{pmatrix} c\partial_\xi & 0 \\ 0 & \partial_{\xi\xi} + c\partial_\xi \end{pmatrix}$$

generates a bounded semigroup. Indeed, the operator $c\partial_\xi$ generates the semigroup $P(t)s(\xi) = s(\xi + ct)$, which is a C^0-semigroup. The operator $\partial_{\xi\xi} + c\partial_\xi$ on E_0 is sectorial (see [20], pp. 136-137, and [29], Section 3.2, Corollary 2.3) and hence generates an analytic semigroup ([20], Theorem 1.3.4). Note that Theorem 9.5 does not apply to the full system since $L^{(1)}$ is not parabolic.

The behavior of the perturbation $\widetilde{m}$ to $\widehat{m}$ mimics the behavior of the perturbation $\widetilde{u}$ to the $\widehat{u}$ component of the front: $\|\widetilde{m}\|_0$ is bounded by a constant defined by the norm of the initial perturbations to the front and $\|\widetilde{m}\|_\alpha$ converges exponentially to 0 as $t \to \infty$. Taking into account exponential convergence of the $\widetilde{y}$ -component to 0 in both norms, we can conclude that $\widetilde{v}$, which is the perturbation to the temperature component, stays bounded (and controlled by the size of the initial conditions) in the $\|\cdot\|_0$-norm and converges to 0 in $\|\cdot\|_\alpha$-norm.

9. Appendix.

9.1. Semigroup Estimates. For our system the semigroup estimate follows from a theorem of [15] described below. Let the linearization of a PDE system about a wave (a front or a pulse) traveling with velocity c have the form

$$\begin{aligned} \partial_t U &= D\partial_{\xi\xi} U + (\widetilde{A} + diag(c, \dots, c))\partial_\xi U + B_{11}(\xi)U + B_{12}(\xi)V, \\ \partial_t V &= c\partial_\xi V + B_{21}(\xi)U + B_{22}(\xi)V, \end{aligned}$$

where U is a $k \times 1$ vector, V a $(n-k) \times 1$ vector, D and $\widetilde{A}$ are constant matrices, $D = \mathrm{diag}(d_1, \dots, d_k)$, all $d_i > 0$, $1 \le k \le n$, and each $B_{ij}(\xi)$ exponentially approaches a constant matrix $B_{ij}^{\pm}$ as $\xi \to \pm\infty$. Consider the linear operator $\mathcal{L}$ associated with the differential expression

$$\mathcal{L} = \begin{pmatrix} D\partial_{\xi\xi} + (\widetilde{A} + diag(c, \dots, c))\partial_\xi + B_{11} & B_{12} \\ B_{21} & c\partial_\xi + B_{22} \end{pmatrix}$$

on one of the Banach spaces $L^2(\mathbb{R})^n$, $H^1(\mathbb{R})^n$, $L^1(\mathbb{R})^n$, or $BUC(\mathbb{R})^n$.

Theorem 9.1. *[15] Suppose the spectrum of $\mathcal{L}$ is contained in $\mathrm{Re}\,\lambda \le -\nu$, $\nu > 0$, except for an eigenvalue at 0 of finite algebraic multiplicity. Let $\mathcal{P}_0$ be the Riesz spectral projection for $\mathcal{L}$ whose kernel is equal to the generalized eigenspace for the 0 eigenvalue, and let $0 < \delta < \nu$. Then there is a number $K > 0$ such that*

$$\|e^{t\,\mathcal{L}\,\mathcal{P}_0}\| \le K e^{-\delta t}.$$

The proof of Theorem 9.1 is based on the Gearhart-Prüss or Greiner Spectral Mapping Theorem, for dealing with Hilbert space and Banach space, respectively.

9.2. Nonlinear Stability. The following theorem is formulated and proved in [14]. Consider the system

$$Y_t = DY_{xx} + R(Y), \tag{9.1}$$

with $x \in \mathbb{R}$, $t \geq 0$, and $Y : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$. The function $R : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, $D = \text{diag}(d_i)$ is an $n \times n$ constant diagonal matrix with $d_i > 0$ for $i = 1, \ldots, k$, where k a number between 1 and n, and $d_i = 0$ otherwise. Let

$$Y_t = DY_{\xi\xi} + cY_\xi + R(Y), \tag{9.2}$$

be the system that has as a solution a front or a pulse Y^* that moves with the velocity c and that asymptotically connects the equilibrium Y_- at $-\infty$ to the equilibrium Y_+ at $+\infty$. Without loss of generality we take Y_- to be a zero vector.

Assume that, in (9.2), Y can be written as $Y = (U, V) \in \mathbb{R}^n$, where $U \in \mathbb{R}^k$, $V \in \mathbb{R}^{n-k}$, and the splitting is such that there exists a $k \times k$ matrix A_1 such that $R(U, 0) = (A_1 U, 0)$, and the equation (9.1) has the form

$$\begin{aligned} U_t &= D_1 U_{\xi\xi} + cU_\xi + R_1(U, V), \\ V_t &= D_2 V_{\xi\xi} + cV_x + R_2(U, V), \end{aligned}$$

where D_1 and D_2 nonnegative diagonal matrices, and R_1 and R_2 are continuously differentiable maps. The linearization of (9.2) then at $Y_- = (0, 0)$, reads

$$\begin{aligned} U_t &= L^{(1)} U + D_V R_1(0,0)V := D_1 U_{\xi\xi} + cU_\xi + A_1 U + D_V R_1(0,0)V, \\ V_t &= L^{(2)} V := D_2 V_{\xi\xi} + cV_\xi + D_V R_2(0,0)V, \end{aligned}$$

Additional assumptions can be formulated as:

Hypothesis 9.2. *Assume that the traveling wave Y_* is spectrally stable in an exponentially weighted space E_α.*

Hypothesis 9.3. *Assume that (i) the operator $L^{(1)}$ on E_0 generates a bounded semigroup, and (ii) the operator associated with $L^{(2)}$ on E_0 has its spectrum in the half-plane $\operatorname{Re} \lambda \leq -\nu$ for some $\nu > 0$.* The following theorem then holds.

Theorem 9.4. *Assume Hypotheses 9.2 and 9.3. Then the wave Y_* is nonlinearly convectively unstable, with α given by Hypothesis 9.2. Thus if the perturbation $(\widetilde{U}(\cdot, 0), \widetilde{V}(\cdot, 0))$ of the traveling wave is initially small in $E_0 \cap E_\alpha$, then the corresponding solution of (9.2) decays exponentially in E_α as $t \to \infty$ to a particular shift of the wave. The solution can be written as*

$$(U, V)(\xi, t) = (U_*(\xi + q(t)) + \widetilde{U}(\xi, t), V_*(\xi + q(t)) + \widetilde{V}(\xi, t))$$

where, for each t, the function $(\widetilde{U}(\cdot, t), \widetilde{V}(\cdot, t))$ belongs to a fixed subspace of $E_0 \cap E_\alpha$ complementary to the the span of $Y_*'(\cdot)$. $\widetilde{U}(\cdot, t)$ stays small in E_0, while $\widetilde{V}(\cdot, t)$ decays exponentially in E_0 as $t \to \infty$, and the function $q(t)$ converges exponentially to a finite limit as $t \to \infty$.

Theorem 9.5. *In addition to the assumptions in Theorem 9.4, let us suppose that the linear equation $U_t = L^{(1)} U$ is parabolic, i.e., the diagonal entries of D_1 are all positive. If the initial perturbation of the traveling wave is also small in L^1-norm, then $\widetilde{U}(\cdot, t)$ stays small in the L^1-norm and decays like $t^{-\frac{1}{2}}$ in the L^∞-norm as $t \to \infty$.*

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